**NATIONAL OPEN UNIVERSITY OF NIGERIA**
**University Village, Plot 91, Cadastral Zone, Nnamdi Azikwe Express Way, Jabi-Abuja**
**FACULTY OF SCIENCES**
**DEPARTMENT OF MATHEMATICS**
**2025_2 EXAMINATIONS**

---

**Course Title: Functional Analysis II**
**Credit Unit: 3**
**Time Allowed: 3 Hours**
**Total: 70 Marks**
**Instruction: Answer Question One (1) and Any Other Three (3) Questions**

**Question One**

**1(a)**Let $X = \mathbb{R}^2$. For arbitrary $\bar{x} = (x_1, x_2)$ in $X$, define the function $\|\cdot\|_2 : \mathbb{R}^2 \to [0, \infty)$ by $\|\cdot\|_2 = (x_1^2 + x_2^2)^{1/2}$ by only showing that this function satisfies the triangle inequality, prove that it defines a norm on $\mathbb{R}^2$

**(b)**Let $X = \mathbb{R}$, the set of real numbers. Show that the map $\rho(x, y) = |x - y|$ defines a metric on$\mathbb{R}$.

**(c)**Every normed space is automatically a metric space. Prove this claim

**(d)(i)** What is a Banach space?

**(ii)**What does it mean to say a normed space is complete?

**(e) (i)** All normed spaces are metric spaces. Does the converse of this statement hold?

**(ii)** Let $X = l_2 := \{x = (x_1, x_2, \cdots) : x_i \in \mathbb{R},\ \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$. Let $x = \left(1, \frac{1}{2}, \frac{1}{3}, \cdots\right)$.

Is $\in l_2$ **(25marks)**

**Question Two**

**(a)**Prove that the space $C[a, b]$ of continous real valued functions defined on the closed and bounded interval [a,b] is complete if endowed with the sup norm $\| f \|_0 = \max\limits_{a \le t \le b} |f(t)|$ for arbitrary $f \in C[a, b]$

**(b)**Let$(X, \rho)$ be a complete metric space, and let $E \subset X$. $(E, \rho_E)$ is complete if and only if it is closed. Where $\rho_E$ is the subspace metric induced by $\rho$. Prove this theorem . **(15marks)**

**Question Three**

**(a)**Let$X$ be a linear space. Define a map $T: X \times X \to X \times X$ by

$$T(x, y) = (x + 2y, x - 2y) \quad ; \quad (x, y) \in X \times X$$

**(i)** Determine if $T$ is linear **(ii)** Does $T$ have an inverse map?

**(b)** Let $C$ denote the linear space of complex numbers over $C$ (as the field of scalars), and define the map $T: \mathbb{C} \to \mathbb{C}$ by $Tz = \bar{z}$, where $\bar{z}$ denotes the complex conjugate of $z$. Prove that $T$ is not a linear map. **(15marks)**

**Question Four**

**(a)**Let$(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Prove that the function $\|\cdot\|: X \to \mathbb{R}$ defined by $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on $X$.

**B (i)** Given an inner product space $(X, \langle\cdot,\cdot\rangle)$, Prove the equality

$\langle z, \lambda x + \mu y\rangle = \bar{\lambda}\langle z, x\rangle + \bar{\mu}\langle z, y\rangle$ for arbitrary vectors $x, y, z \in X$ and scalars $\lambda, \mu \in \mathbb{C}$

**(ii)** What is the usefulness of the parallelogram law?

$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ ? **(15marks)**

**Question Five**

**(a)(i)** What is a Hilbert space? **(ii)** Given an inner product space $(X, \langle\cdot,\cdot\rangle)$ and a set $S \subset X$, when is $S$ said to be Orthonormal? **(iii)** What is a complete orthonormal set?

**(b)**Let$\mathbb{H}$ be a real Hilbert space and let $M$ be a closed subspace of $\mathbb{H}$. Consider an arbitrary vector $x \in \mathbb{H}$ . State three major questions answered by the projection mapping theorem

**(c)(i)** Let $M$ and $N$ be arbitrary subspaces of a Hilbert space $\mathbb{H}$.

Prove the following $M \subset M^{\perp\perp}$

**(ii)** State the Riesz Representation Theorem**(15marks)**

**Question Six**

**(a)**Let$\mathbb{H}$ be a Hilbert space.When is an operator $T: \mathbb{H} \to \mathbb{H}$ said to be

**(i)** Self-adjoint **(ii)** Normal **(iii)** Unitary

**(b)(**Consider the right shift operator on $l_2$ , $T: l_2 \to l_2$ defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$$

**(i)** Find the adjoint of this map **(ii)** Is this map normal? **(15marks)**